The Matrix-Tree Theorem

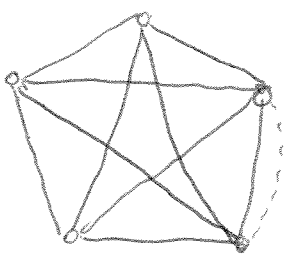
- Lots of cool ideas
- Lots of directions to go in

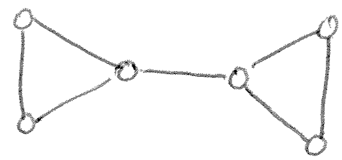
$\Big\}$ Why I'm giving this talk.

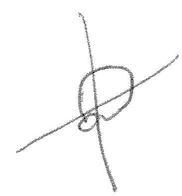
  - Combinatorics - Critical groups, ~~&~~ etc.
  - Spectral Graph Theory

Def: Graph - set V of n vertices
E edges between vertices.

Ex.



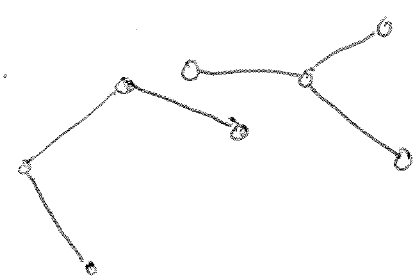




For this talk:
- G - graph w/ n vertices, e edges.
- Mult. edges allowed, no self loops.

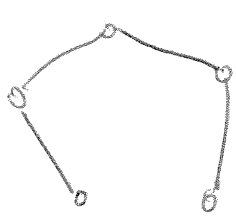
Def: ~~Spanning~~ Tree
Graph w/ no cycles.

Ex.



Def: Spanning Tree for G.
Subgraph of G which is a tree, connecting every vertex.

Ex.



Lemma: T subgraph of G,
T is a spanning tree iff T has $n-1$ edges, no cycles.

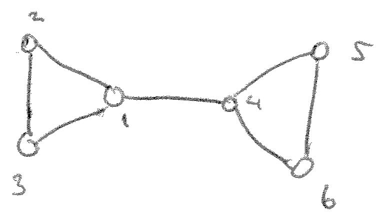
Pf: Seen.

Def: Laplacian of a graph G. – $n \times n$ matrix.

$$L_{ij} = \begin{cases} \deg(v_i) & \text{if } i=j \\ -m_{ij} & \text{if } i \neq j \end{cases}$$

($\deg(v_i)$ = # edges touching v_i
m_{ij} = # edges from i to j.)

Ex.

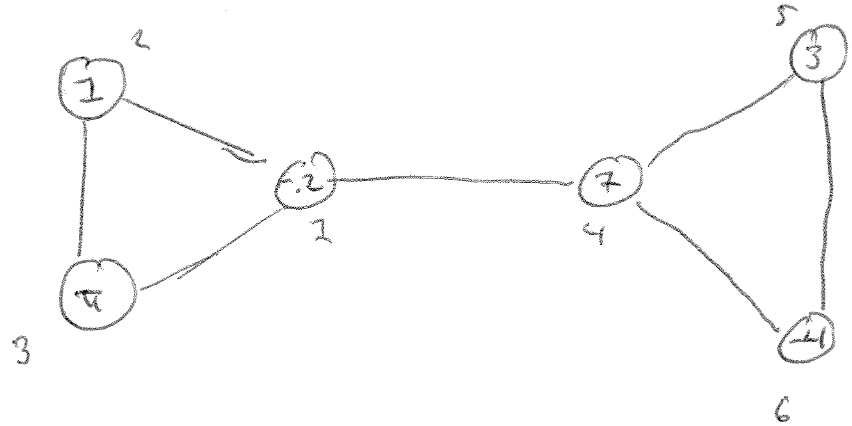


$$\rightsquigarrow L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Rmk: L is symmetric i.e. $L = L^T$.

$L : \mathbb{R}^n \to \mathbb{R}^n$

think of this as functions on the vertices of G.



$$\rightsquigarrow \begin{bmatrix} .2 \\ 1 \\ \pi \\ 7 \\ 3 \\ -4 \end{bmatrix}$$

f: function on vertices, $f(v_i)$ = value at vertex i.

Q: What is Lf?

$$(Lf)(v_i) = \deg(v_i)\, f(v_i) - \sum_j m_{ij}\, f(v_j).$$

Q: When is $f \in \operatorname{Ker} L$?

$$0 = \deg(v_i)\, f(v_i) - \sum_j m_{ij}\, f(v_j)$$

$$\rightsquigarrow\quad f(v_i) = \frac{1}{\deg(v_i)} \sum_j m_{ij}\, f(v_j)$$

So: $f \in \operatorname{Ker} L$ if $f(v_i)$ = avg. value of f @ neighbors of v_i.

(functions w/ this property are called harmonic).

~~So: .... f = const $\in$ Ker L.~~

Warm up: What can we say about rank(L)? Can L be invertible?

No: $f(v_i) = 5 \in$ Kernel. (any constant).

If G has 2 conn. comp.

G = 

consts here + consts here $\in$ Ker L.

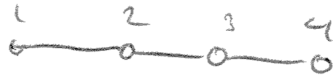
$\Rightarrow$ rank(L) $\leq n - \#$ Conn. comp. of G.

Matrix-Tree Thm: (Kirchoff's Thm)

Let G be a graph w/ no self-loops, $k(G)$ be the number of spanning trees, and $L(G) = L$ the laplacian. Let $L_0 = $ ~~Laplacian~~ w/ last row/column removed.

Then $\det(L_0) = k(G)$

Ex. 1)



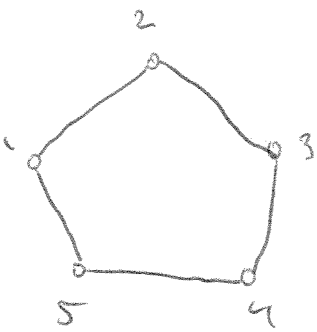
$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \rightsquigarrow L_0 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\cancel{\det(L_0) \overset{?}{=} 1}$

$k(G) \overset{?}{=} 1$

row reduce $\rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \det 1$

Ex 2



$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$\cancel{\det(L_0) \overset{?}{=}}$

$k(G) \overset{?}{=} 5$

$$L_0 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

$\det(L_0) = 5$. ✓. Cool!!

Lemma: Cauchy-Binet Thm:

A - $m \times n$ matrix B - $n \times m$ matrix, $m \leq n$

~~If $m \leq n$~~ then

$$\det(AB) = \sum_{\substack{S \subseteq [n] \\ |S| = m}} \det(A[S]) \det(B[S])$$

$[n] = \{1, \ldots, n\}$

$A[S]$ = S - colums of A

$B[S]$ = S - rows of B.

Ex $A = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} = B$

$m = 2$, $n = 4$

$S = \{2, 4\}$.

Give G an arbitrary orientation.

($\circ \to \circ$ - orient edges)

M - incidence matrix $M_{ij} = \begin{cases} -1 & \text{if } e_j \text{ starts @ } v_i \\ 1 & \text{if } e_j \text{ ends @ } v_i \\ 0 & \text{else.} \end{cases}$

Note: • colums indexed by edges

• every column has exactly • 1 entry $+1$

• 1 entry -1

rest are 0.

Rmk: This is the boundary operator

Lemma: $L = M M^T$

$$(MM^T)_{ij} = \sum_{\substack{k \\ e_k \text{ edge}}} M_{ik} M^T_{jk} = \sum_{\substack{k \\ e_k \text{ edge}}} M_{ik} M_{jk}.$$

$i \neq j$: $M_{jk} M_{ik} = \begin{cases} 0 & e_k \text{ doesn't touch } v_i \text{ or } v_j. \\ -1 & \text{if } k \text{ btw } v_i + v_j \end{cases}$

$\Rightarrow \sum_k M_{ik} M_{jk} = -m_{ij}$ (why?)

$i = j$. $M_{ik}^2 = \begin{cases} 1 & \text{if } e_k \text{ touches } v_i. \end{cases}$

$\Rightarrow \sum_k M_{ik}^2 = \deg(v_i)$. $\square$

Set $M_0 = M$ w/ last row removed.

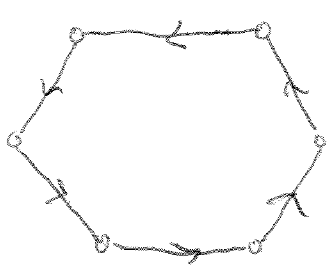
Lemma: S set of $n-1$ edges of G.

1) If S not a spanning tree, then $\det(M_0[S]) = 0$

2) If S is a spanning tree, $\det(M_0[S]) = \pm 1$

Pf: If S not a spanning tree

$\Rightarrow S$ has a cycle, $e_1 \ldots e_s$.



Goal: Produce a linear relation among columns for e_i.

Mult. columns by -1 to "re-orient" edges.

Then sum of all of them is zero.

(Why?)

Suppose S is a spanning tree

Let e_ℓ be an edge connected to v_n

(exists b/c spanning tree; last row we removed)

Let v_k be other vertex.

e_ℓ - column has 1 non-zero entry

~~use it to column eliminate~~

~~Cofactors~~

$M(S')$ = matrix for a tree S' w/ e_ℓ contracted to a vertex $\tilde{v}$

$M_0'(S')$ = matrix for S' w/ row for $\tilde{v}$ removed

$\det(M_0^t[S]) = \pm \det(M_0'[S'])$

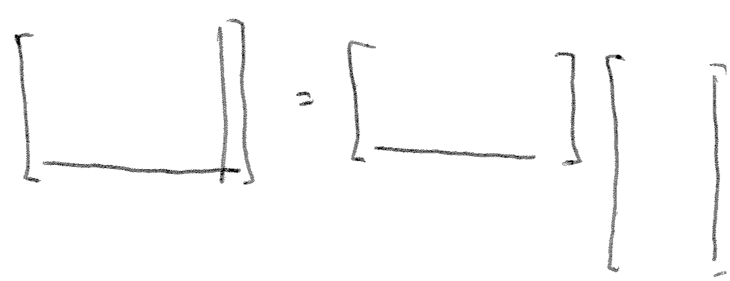
by cofactor expansion

Induction $\Rightarrow$ done (check for)

<u>Pf of Thm:</u> $L = M M^T$

L

$\Rightarrow L_0 = M_0 M_0^T$



Cauchy-Binet $\Rightarrow$ $\det(L_0) = \sum_{|S|=n-1} \det(M_0[S]) \det(M_0^T[S])$

$= \sum_{|S|=n-1} \det(M_0[S])^2 = k(G)$

Some Spectral Graph Theory:

L - symmetric, i.e. $L = L^T$

Spectral Thm: $\Rightarrow$ L has orthonormal basis of eigenvectors.

use eigenvalues / eigenvectors to get info about graph.

Ex) A - adjacency matrix

$$A_{ij} = m_{ij} = \#\text{edges from } i \text{ to } j.$$

- symmetric, Let μ_1 be the largest eigenvalue.

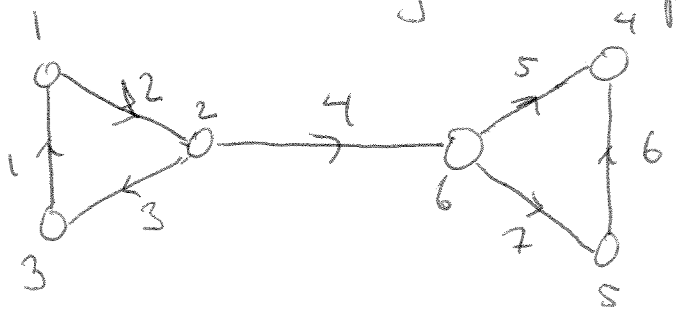
Then: $d_{ave} \leq \mu_1 \leq d_{max}$

and $\chi(G)$ = chromatic number of G.

$$\Rightarrow \quad \chi(G) \leq \lfloor \mu_1 \rfloor + 1$$

Ref: Dan Spielman's notes:

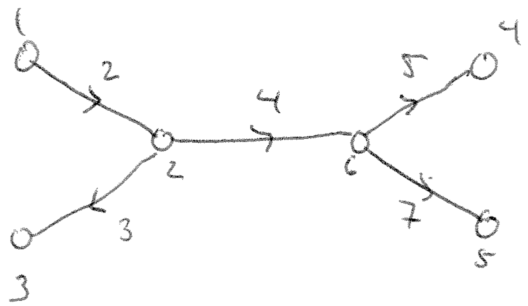
Running Example.



$$M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cancel{0}\,-1 & -1 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cancel{1}\,0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cancel{0}\,-1 & -1 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cancel{1}\,0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

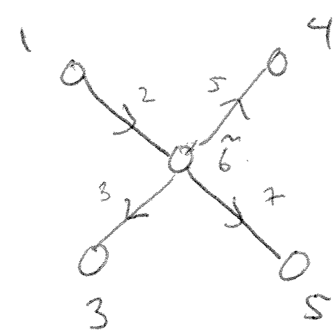
__Ex.__ $S = \{2, 3, 4, 5, 7\}$.



$v_n = v_6$

$e_l = e_4$

$v_k = v_2$.



T'

$$M_0[S] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & \cancel{0}\,-1 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \qquad M'_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(M_0[S]) = \pm \det(M_0'[S'])$$